Principles of design of projections for geographical information technologies

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Abstract. Representation of geographic data in a computer and subsequently on papers requires access to mechanisms to project the curved earth surface onto a plain medium (computer screen or paper). There is a large number of known projections. However, it is sometimes difficult to choose a proper projection for the nonexpert. In this paper we present a library of cartographic projections that can be used in GIS. We have put special emphasis on composite projections.

1 Introduction

The requirements to the coordinate base of geographical information technologies are formulated based on a general algorithm for a class of conformal transforms of the ellipsoid's surface on a plane. A specific projection type is defined by one equation that describes the representation of the central meridian on the plane. A new class of projections, obtained by linear composition of conical and cylindrical projections, is suggested.

We present a library of cartographic projections developed by us. We suggest that this library should be used in modern software packages for representation of spatial information in a certain coordinate system. The library is constantly being extended with software improvement and development. Certain knowledge about projections is required of users and creators of automated informational systems, in order to solve tasks for a specified territory and select a relevant projection from the library. However, this can present certain problems.

Nowadays in our opinion the task of determining the class of projections for geographical information technologies that must combine the best characteristics of both cartographic and geodetic projections in their conventional understanding is very important. The advances of modern, computer-based measuring and processing technologies in geodesy, cartography and photogrammetry, including methods of data acquisition, transmission and presentation, allow their efficient use for different purposes. We intend to develop a mathematically determined technology for representation of different spatial information over territories of different size and boundary configuration. The best way would be to represent spatial objects in the most convenient for the majority of users coordinate system. This can be the system of plane rectangular coordinates obtained in the projection of ellipsoid on the plane.

From our point of view the parameters for such a projection should be selected automatically using a computer. They should meet the following requirements:

- a general mathematical description of the algorithm should exist, to ensure the required accuracy of calculation for a certain class of projections. This description should be based on the theory of mutual representation of the terrestrial ellipsoid surface and the plane;
- the possibility to ensure reliable correspondence with state geodetic coordinate systems and with other coordinate systems that might be used in GIS.
- the projection shoud be readily available, simple and convenient to use for various user categories;
- ensure minimal, and possibly negligible, small distortions of the objects and their relative position;

2 Development of projection

We have developed a general theory to describe a certain class of conformal projections. We have also developed a general algorithm to calculate their numerical characteristics with the help of a computer [2], [3]. This class of projections considers most common geodetic projections as special cases (Gauss-Krüger cylindrical projection, Lambert conical projection, Russilhe stereographic projection [1]). This class of projections allows us to generate new projections, to ensure minimal possible distortions for the geometrical shapes present. In the general algorithm we have selected the equation that expresses the image shape on the meridian plane of the ellipsoid in the form of a decreasing series, in accordance with the amount of change inside the area of the isometric latitude Δq. It also determines the shape of the isocols (the contours of constant distortion) and, consequently, the type and characteristics of the distribution of distortion inside this area. We call this equation the characteristic projection equation.

It has the following general form:

$$\Delta X = X - X_0 = \sum_{j=1}^{n} c_j \Delta q^{(n)} = \sum_{j=1}^{n} c_j P_1^{(n)} \qquad (1)$$

Calculations are made according to the following formulas:

for the relation of ellipsoid coordinates (isometric latitude q and longitude L) and plane rectangular coordinates x, y;

$$\Delta x = x - x_0 = \sum_{j=1}^{n} c_j P_j; \qquad \Delta y = y - y_0 = \sum_{j=1}^{n} c_j Q_j, \tag{2}$$

$$\Delta q = q - q_0 = \sum_{j=1}^{n} c_j^/ P_j^/; \qquad \Delta L = L - L_0 = \sum_{j=1}^{n} c_j^/ Q_j^/, \tag{3}$$

where x_0, y_0, q_0, L_0 are the coordinates of the initial point of the projection. x_0 *and* y_0 can be set equal to the length of the meridian arc and the ellipsoid parallel from the equator and the Greenwich meridian correspondingly, or to any other fixed numerical value depending on the current task.

Here c_j, are the coefficients of the characteristic projection equation, and $c_j^/$ are the coefficients of the inverted power series (1)

$P_j = P_1 P_{(j-1)} - Q_1 Q_{(j-1)}$; $Q_j = Q_1 P_{(j-1)} - P_1 Q_{(j-1)}$;
$P_1 = \Delta q = q - q_0$,, $Q_1 = \Delta L = L - L_0$; $P_0 = 1$, $Q_0 = 0$;
$P^/_j = P^/_1 P^/_{(j-1)} - Q^/_1 Q^/_{(j-1)}$; $Q^/_j = Q^/_1 P^/_{(j-1)} - P^/_1 Q^/_{(j-1)}$;
$P^/_1 = \Delta x = x - x_0$,, $Q^/_1 = \Delta y = y - y_0$; $P^/_0 = 1$, $Q^/_0 = 0$;

- the local scale lengths m and and the grid convergence on the plane γ are calculated by the equations (4),

$$m = \frac{\sqrt{K_1^2 + K_2^2}}{r}, \qquad \gamma = arctg\left(\frac{K_1}{K_2}\right) \tag{4}$$

under the assumption that

$$K_1 = -\sum_{j=1}^{n} j c_j Q_{(j-1)}; \qquad K_2 = \sum_{j=1}^{n} j c_j P_{(j-1)} \tag{5}$$

Schols's equation for calculating the curvature of the representation of the ellipsoid's geodetic line on the plane

$$\Gamma = \Gamma_1 \sin A - \Gamma_2 \cos A; \tag{6}$$

(For more detail see Appendix A1, Schols's equation).

In the characteristic equations for projections of this class we provide the possibility to vary the values of the local scale length $m_0 \le 1$ on the central meridian (in transverse cylindrical projections), on the standard parallel (in conical projections), in the central point (in stereographic, azimuthal and composite projections).

For instance, the coefficients c_j of the characteristic equation for conical projections are expressed as

$$c_j = \frac{c_1}{j!}(-1)^{(j-1)}(\sin B_0)^{(j-1)} \qquad (7)$$

Note that the first two coefficients of all projections of this class are the same and are expressed as

$$c_1 = m_0 r_0; \qquad c_2 = -\frac{c_1}{2}\sin B_0 \qquad (8)$$

where r_0 is the radius of the parallel at the latitude of the central point of the projection and B_0 is the geodetic latitude of the central point of the represented area.
For transversal cylindrical projections (Gauss-Krüger, *UTM*) we provide the coefficients of the characteristic equation up to the eighth degree inclusive (appendix A2).
Coefficients of the characteristic equation for Russilhe and Gauss stereographic projections in a form analogous to (8) up to the eighth degree inclusive have also been derived.
Analogous formulae have been developed for the reverse transform when it is necessary to change over from the elements on the projection plane to the relevant elements on the ellipsoid's surface. Here coefficients B'_j of the inverted power series are involved. The coefficients can be obtained for every type of projection of this class. However, for calculations with a computer it is more efficient to have general expressions of a type up to the eighth degree inclusive. (*cf.* Appendix A3)
In Lambert's conical projections for coefficients B'_j there is a general expression for any n

$$c'_j = \frac{(\sin B_0)^{(j-1)}}{j(c_1)^j} \qquad (9)$$

3 Discussion

To minimize the representation distortion on the surface of a region of the ellipsoid with arbitrary size and boundary shape, provisions are made in the general algorithm to handle the following possibilities:

selection of a type of projection that has isocols (lines of equal distortion) close to the shape of boundaries of the represented area. In this case we obtain a projection that satisfies the Chebyshev-Grave criterion of the best projection;

selection of a value for the local scale of length m_0 in the initial point of the projection allows to adjust the distribution of distortions within the area represented.

As is well known in general, no individual projection can have isocols of an appropriate shape.

The first possibility is fulfilled with a computer by getting new projections based on the linear combination of coefficients of characteristic equations of known projections of a suggested class. According to the research results it is sufficient to have only conical and cylindrical conformal projections, e.g. of Gauss-Krüger and Lambert. As the composition of coefficients of conformal projections is linear, we get a composite projection that is also conformal because of the compliance with the Cauchy-Riemann conditions

$$k_1 = k_2$$

This is essential for the practical implementation of the general algorithm to calculate projections of the suggested class, since linear distortions in conformal projections are, due to the scale of representation, always more significant as compared to the influence of the distortion due to the curvature of representation of an ellipsoid geodetic line on the plane.

If we have the expansion coefficients $C_{j\,(1)}$ and $C_{j\,(2)}$ (see equ. (1)) for cylindrical and conical projection respectively, then the coefficients of the composite projection can be obtained using the following formula

$$C_j = k_1 C_{j(1)} + k_2 C_{j(2)} \ , \qquad (10)$$

where composition coefficients characterize the degree of the projections involvement in the composition and must satisfy

$$k_1 + k_2 = 1 \ . \qquad (11)$$

Coordinates in the composite projection will differ from coordinates in the projections involved in the process within small magnitudes of the third order.

The value of one of the composite coefficients may be arbitrary, but one will get projections with different shapes of isocols:

$k_1 = 1$ – cylindrical projection, isocols are parallel and symmetrical to the image of the central meridian;

$k_1 = 0$ – conical projection, isocols are parallel and almost symmetrical to the image of the standard parallel;

$k_1 = 0.5$ – Gauss stereographic projection (a special case of azimuthal projection), isocols are close to the circle which is circumscribed around the central point of projection;

$k_1 < 0$ or $k_2 < 0$ – isocols are represented by the class of two pairs of conjugate hyperbolas and their asymptotes.

$(k_1 \neq k_2) > 0$ – isocols are close to the shape of an ellipse elongated along the central meridian when $k_1 > k_2$ and along the standard parallel when $k_1 < k_2$

The general equation describing the isocols for a given class of projections is as follows

$$\frac{k_1x^2 + k_2y^2}{2m_0(m - m_0)R_0^2} = 1 \qquad (16)$$

The second possibility of the general algorithm is fulfilled by the way similar to that of *UTM* projection. Here a simple formula used for any type of projections is applied

$$m_0 = \frac{2}{1 + m_{\max}} \qquad (17)$$

Here m_{max} is the maximal scale value within the imaged area when $m_0 = 1$. One can notice that $m_0 = 0.9996$, assumed for the *UTM* projection, is optimal for a 6-degree coordinate zone of Gauss-Boaga cylindrical projection, which has linear shape for average latitude of USA. Naturally, for other countries this value will be different. After the optimal value for scale at the initial point of the projection has been choosen it is possible to get just a half of the maximum distortion. For example, for Gauss projection, when $m_0 = 1$ for an average latitude of USA at the edge of a 6-degree zone, the distortion of lengths can reach a magnitude of 1 : 1250. For the *UTM* projection, when $m_0 = 0.9996$, the magnitude of distortion is 1 : 2500. Distortions at the central meridian and at the edge of the zone in such case are equal with regard to the absolute magnitude. The maximum change of scale inside the zone is the same.

If according to the task one needs to obtain negligibly small distortions for the same areas inside the initial area, in particular, one needs to depict boundaries of the object without distortion, one must use the most suitable projection for this territory representation and choose isocols $m = const$, which goes along this area and the scale at the initial point is calculated according to the following formula

$$m_0 = \frac{1}{m}$$

4 Conclusions

Automated realization of the algorithm for forming the coordinate system for geographical information technologies is not difficult. The general algorithm allows making calculations of linear values with accuracy not less than 0.001m and of angular values $0.001''$. To achieve this the area in question should not exceed 12° in latitude and longitude. The coefficients have a constant value within the represented area.

When we have lower accuracy requirements on the geometric images, the size of the area represented in one coordinate zone increases. Selection of the projection that will ensure least possible distortions in GIS is very important. This reduces the volume and complexity of geodetic measurements and, what is more important, geometric images' characteristics (distances, areas, etc.) calculated in plane rectangular coordinates have less differences from their analogues on the earth surface. In most cases these differences are negligibly small.

It is clear that in any case distortions of the regions of ellipsoid surface represented on the plane depend on its area. The smaller the area is, the smaller may distortions be. For geographical information technologies the precision of the representation of spatial information depends on commonness of a task to be solved (state, regional or local level). The suggested solution for designing coordinate systems for geographical information technologies allows obtaining projections with minimal possible distortions, the maximum values of which depend on the area of the represented region and are practically independent of the shape of its boundaries.

The suggested method allows us to minimize the distortions for regions of the surface of ellipsoids of different size and shape. On the basis of this method we can form coordinate systems interconnected by the common algorithm. This ensures the possibility of including separate fragments into a general system, extraction of separate elements from the general system as well as their more precise representation (with less distortions) in the system of plane rectangular coordinates.

References

1 Bugayevskiy, L.M. and J.P. Snyder, 1995. Map Projections – A reference manual. Taylor & Francis: London.

2 Padshyvalau U. P. 1997. *Coordinate environment for GIS.*. Geodesy and Cartography, №6, p. 51 – 55, Moscow

3 Padshyvalau U. P. 2000. *Composite geodesic projections*. Geodesy and Cartography, №8, p. 39 – 43, Moscow.

Appendices

A1 Details of Schols's equation

Schols's equation to calculate the representation of the curvature of the geodetic line of the ellipsoid on the plane

$$\Gamma = \Gamma_1 \sin A - \Gamma_2 \cos A; \qquad \text{(A1)}$$

where it is assumed that:

$$\Gamma_1 = \frac{\left(K_1^2 + K_2^2\right)\sin B + \left(K_1 K_3 - K_2 K_4\right)}{\left(K_1^2 + K_2^2\right)^{3/2}}; \qquad \Gamma_2 = \frac{K_1 K_4 + K_2 K_3}{\left(K_1^2 + K_2^2\right)^{3/2}},$$

where A s the geodetic azimuth of the line, and B is the geodetic latitude of the point,

$$K_3 = -\sum_{j=1}^{n} j\left(j+1\right)c_{(j+1)}Q_{(j-1)}; \qquad K_4 = -\sum_{j=1}^{n} j\left(j+1\right)c_{(j+1)}P_{(j-1)}$$

- general equations for the relation of polar coordinates on the plane and ellipsoid:
  for distances

$$S_{Plane} = S_{Ellipse}\left(\frac{m_i + m_k + 4m_m}{6} - \Gamma_m^2 \frac{S^2}{24}\right) \qquad \text{(A2)}$$

for direction angles and geodetic azimuths

$$\alpha_{ik} = A_{ik} - \gamma_i + \Gamma_i \frac{S^2}{2} \qquad \text{(A3)}$$

A2. Coefficients of characteristic equation for transversal projections

$$c_1 = m_0 r_0; \qquad c_2 = -\frac{c_1}{2}\sin B_0; \qquad c_3 = \frac{c_1}{12}\cos^2 B_0\left(2tg^2 B_0 - 1 - \eta_0^{\prime 2}\right)$$

$$c_4 = \frac{c_1}{24}\sin B_0 \cos^2 B_0\left(2 - tg^2 B_0 + 6\eta_0^{\prime 2} + 4\eta_0^{\prime 4}\right)$$

$$c_5 = \frac{c_1}{240}\cos^4 B_0\left(2 - 11tg^2 B_0 + 2tg^4 B_0 + 12\eta_0^{\prime 2} - 91\eta_0^{\prime 2} tg^2 B_0\right) \qquad (11)$$

$$c_6 = \frac{c_1}{1440}\sin B_0 \cos^4 B_0\left(26tg^2 B_0 - 17 - 2tg^4 B_0 - 270\eta_0^{\prime 2} + 570\eta_0^{\prime 2} tg^2 B_0\right)$$

$$c_7 = \frac{c_1}{20160}\cos^6 B_0\left(180tg^2 B_0 - 17 - 114tg^4 B_0 + 4tg^6 B_0\right)$$

$$c_8 = \frac{c_1}{40320}\sin B_0 \cos^6 B_0\left(62 - 192tg^2 B_0 + 60tg^4 B_0 - tg^6 B_0\right)$$

Here it is assumed that $\eta_0^{\prime 2} = e^{\prime 2} cos^2 B_0;$ *where* $e^{\prime 2}$ is the second eccentricity of the meridian ellipse.

A3 Coefficients for a reverse projective transform

$$
\begin{aligned}
&c_1^{/} = \frac{1}{c_1}; \qquad c_2^{/} = -\frac{c_2}{c_1^3}; \qquad c_3^{/} = \frac{1}{c_1^5}\left(2c_2^2 - c_1c_3\right) \\
&c_4^{/} = \frac{1}{c_1^7}\left(5c_1c_2c_3 - c_1^2 - 5c_2^3\right) \quad c_5^{/} = \frac{1}{c_1^9}\left(6c_1^2c_2c_4 + 3c_1^2c_3^2 + 14c_2^4 - c_1^3c_5 - 21c_1c_2^2c_3\right) \\
&c_6^{/} = \frac{1}{c_1^{11}}\left(7c_1^2c_2c_5 + 7c_1^3c_3c_4 + 84c_1c_2^3c_3 - c_1^4c_6 - 28c_1^2c_2^2c_4 - 28c_1^2c_2c_3^2 - 42c_2^5\right) \\
&c_7^{/} = \frac{1}{c_1^{13}}(8c_1^4c_2c_6 + 8c_1^4c_3c_5 + 84c_1c_2^3c_3 + 4c_1^4c_4^2 + 120c_1^2c_2^2c_4 - 28c_1^2c_2^3c_4 + 180c_1^2c_2^2c_3^2 + \\
&+132c_2^6 - c_1^5c_7 - 36c_1^3c_2^2c_5 - 72c_1^3c_2c_3c_4 - 12c_1^3c_3^3 - 330c_1c_2^4c_3); \\
&c_8^{/} = \frac{1}{c_1^{15}}(9c_1^5c_2c_7 - c_1^6c_8 - 45c_1^4c_2^2c_6 - 90c_1^4c_2c_3c_5 - 45c_1^4c_2c_4^2 + 495c_1^3c_2^2c_3c_4 + 165c_1^3c_2^3c_5 - \\
&-495c_1^2c_2^4c_4 + 165c_1^3c_2c_3^3 - 990c_1^2c_2^3c_3^2 + 1287c_1c_2^5c_3 - 45c_1^4c_3^2c_4 + 9c_1^5c_3c_6 + \\
&+9c_1^5c_{4}c_5 - 429c_2^7)
\end{aligned}
\qquad (12)
$$